ON GROWTH AND TORSION OF GROUPS

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Abstract. We give a subexponential upper bound and a superpolynomial lower bound on the growth function of the Fabrykowski-Gupta group.

As a consequence, we answer negatively a question by Longobardi, Maj and Rhemtulla [LMR95] about characterizing groups containing no free subsemigroups on two generators.

1. Introduction

Fabrykowski and Gupta constructed in 1985 a group of intermediate word growth, producing in this way a new example after Grigorchuk's original construction [Gri83].

This group appears originally in [FG85], and is studied further in [FG91]; some of its algebraic properties are explained in [BG02]. A proof of its intermediate growth was first given in [FG85], with an explicit upper bound. However, a gap in the argument lead to a second proof, in [FG91], this time with no upper bound.

Although that second paper's general strategy is sound, many details are missing or incorrect, and we hope to present here the first complete proof. We also give explicit upper and lower bounds on the growth function.

Let us say that two functions f, g satisfy the relation $f \precsim g$ if there is a constant $A > 0$ such that $f(n) \leq g(An)$. We prove the

Theorem 1. *The growth of the Fabrykowski-Gupta group is intermediate. More precisely, if $\gamma(n)$ denote the number of elements expressible as a product of at most n generators of the Fabrykowski-Gupta group, then*

$$e^{n^{\frac{\log 3}{\log 6}}} \precsim \gamma(n) \precsim e^{\frac{n(\log\log n)^2}{\log n}}.$$

We then apply this result to a question by Longobardi, Maj and Rhemtulla. Let G be a group with an exact sequence $1 \to N \to G \to P \to 1$, where N is locally nilpotent and P is periodic. Then G has no free subsemigroup. Indeed, let $x, y \in G$. Then $x^n, y^n \in N$ for some n large enough, so that $\langle x^n, y^n\rangle$ is nilpotent. Hence, neither $\langle x^n, y^n\rangle$ nor $\langle x, y\rangle$ are free as semigroups. (Note that, without loss of generality, one may assume that G is finitely generated).

In [LMR95], Longobardi, Maj and Rhemtulla asked whether the converse were true:

Question 2. *Let G be a finitely generated group with no free subsemigroups. Is G a periodic extension of a locally nilpotent group?*

The answer turns out to be negative; indeed, Ol'shanskii and Storozhev construct in [OS96] a semigroup identity whose free group is not even a periodic extension of a locally soluble group.

We remark that a very simple answer can be given to Question 2, knowing that the Fabrykowski-Gupta group has intermediate growth:

Theorem 3. *The Fabrykowski-Gupta group is generated by two elements, contains no free subsemigroup, and is not a periodic extension of a locally nilpotent group.*

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Proof. Consider a short exact sequence $1 \to N \to G \to P \to 1$, with P periodic. Since G is not periodic (as it contains the element at of infinite order), we have $N \neq 1$. Since G is just infinite, N is of finite index in G, and hence, N is finitely generated. Therefore, as G has intermediate growth, so does N. In particular, N is not (locally) nilpotent. ∎

This example is quite different from the Ol'shanskii-Storozhev example: it is a concrete, residually-3 group which does not satisfy any identity.

2. Settings

2.1. **The Fabrykowski-Gupta group.** Consider the cyclic group of order three $A = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ with generator a, and the 3-regular rooted tree $\mathcal{T}_3 = A^*$, with root $\emptyset$. The automorphism group of A^* is recursively defined by $Aut(A^*) = Aut(A^*) \wr Sym(A)$, and every automorphism decomposes via the map

$$\phi\colon Aut(A^*) \to Aut(A^*) \wr Sym(A);\ f \mapsto \langle\langle\, f_0, f_1, f_2\,\rangle\rangle\sigma$$

where $f_i \in A^*$ and $\sigma \in Sym(A)$. Thus, a acts on $\mathcal{T}_3$ as a cyclic permutation of the first level A of the tree. Define the automorphism t recursively by $t = \langle\langle\, a, 1, t\,\rangle\rangle$. Note that both a and t are of order 3. The group G generated by a and t is called the Fabrykowski-Gupta group. It is known to be a just infinite group, regular branched over G' (see [BG02]).

We still call ϕ the decomposition map of G

$$\phi\colon G \hookrightarrow G \wr A;\ g \mapsto \langle\langle\, g_i\,\rangle\rangle_{i\in A}\,\sigma.$$

Let $Stab(n)$ be the subgroup of G that stabilizes the n^{th} level of A^*. Then $G = Stab(1) \rtimes \langle a\rangle$. Furthermore, $Stab(1) = \langle t\rangle^{\langle a\rangle}$. Let $t_0 = t = \langle\langle\, a, 1, t\,\rangle\rangle$, $t_1 = t^a = \langle\langle\, t, a, 1\,\rangle\rangle$ and $t_2 = t^{a^2} = \langle\langle\, 1, t, a\,\rangle\rangle$ be the generators of $Stab(1)$. Every word $w = w(a^{\pm1}, t^{\pm1})$ uniquely decomposes as

$$w = t_{c_1}^{\gamma_1}\, t_{c_2}^{\gamma_2} \cdots t_{c_n}^{\gamma_n}\, \tau,\ \text{with}\ \gamma_i \in A - \{0\},\ c_i \in A,\ c_i \neq c_{i+1},\ \text{and}\ \tau \in A, \tag{2.1}$$

so that the decomposition map ϕ is defined without ambiguity on the set W of all such words.

We define a word metric on G by assigning the following weights on the generators of G : $\ell(t^{\pm1}) = 1$ and $\ell(a^{\pm1}) = 0$. Then the length of a word $w \in W$, decomposed as in (2.1) is $\ell(w) = n$. That is, the length of w is the number of letters "$t^{\pm1}$" that appear in w. The induced metric on G is

$$\ell(g) = \min\{\ell(w) | w =_G g\},$$

for every $\gamma \in G$. We then define a minimal-length normal form $G \to W; g \mapsto w$ on G.

Note that $\sum_{i\in A} \ell(g_i) \leq \ell(g)$ for every $g \in G$. We will say that $g \in G$ admits *length reduction* if there is a d such that

$$\sum_{i\in A^d} \ell(g_i) < \ell(g),$$

where the g_i's are the states of g on the d^{th} level of the tree (*i.e.*, the components of $\phi^d(g)$).

3. Subexponential growth of fractal groups

A "traditional" way (introduced by Grigorchuk in [Gri84]) to prove that a fractal group G has subexponential growth is to show that every group element admits a fixed proportion of length reduction. More explicitly,

Proposition 4. [BP06] *Let G be a fractal group acting on a d-regular tree, with a word metric ℓ. If there exist constants $0 \leq \eta < 1$ and $k \geq 0$ such that, for the natural embedding $\phi\colon Stab(1) \hookrightarrow G^d : g \mapsto \langle\langle g_1, \ldots, g_d \rangle\rangle$,*

$$\sum_{i=1}^{d} \ell(g_i) \leq \eta\ell(g) + k$$

for every $g \in Stab(1)$, then G has subexponential growth.

3.1. **Length reduction and subexponential growth.** Let G be a finitely generated fractal group acting on a d-regular rooted tree, and let ℓ be a proper seminorm on G. Suppose that for every $g = \langle\langle g_1, \ldots, g_d \rangle\rangle\sigma$ in G, we have $\sum_{i=1}^{d} \ell(g_i) \leq \ell(g)$.

Let $\mathcal{I}_n$ be the subset of G of elements that have no length reduction up to the n^{th} level of the tree. It is defined recursively by $\mathcal{I}_0 = G$ and

$$\mathcal{I}_n = \left\{ g \in G \mid \sum_{i=1}^{d} \ell(g_i) = \ell(g) \text{ and } g_i \in I_{n-1} \text{ for every } 1 \leq i \leq d \right\}.$$

Then, $\mathcal{I} := \bigcap_{n \geq 0} \mathcal{I}_n$ is the set of words that have no length reduction on any level of the tree.

Proposition 5. *Let $G = \langle X \rangle$ be a group as above, with X finite, and $X \subset \mathcal{I}$. If there exists some k such that $\mathcal{I}_k$ has subexponential growth, then G has subexponential growth.*

Moreover, if $\mathcal{I}_k$ has linear growth, then the growth of G is bounded in the following way:

$$\gamma(n) \lesssim e^{n \frac{(\log\log n)^2}{logn}},$$

where $\gamma(n) = \#\{g \in G | \ell(g) \leq n\}$.

Remark. The idea behind this result is the following: if $\mathcal{I}$ grows subexponentially, then, expressing any group element g of length n as a word in $\mathcal{I}^m$ for some m, either m is much smaller than n, and thus the set of such words grows slowly; or m is not negligible compared to n and, in that case, g behaves as in Proposition 4. This kind of argument was used (among other works) in [Bar03]. Anna Erschler has obtained in [Ers04] some similar upper bounds.

In order to prove Proposition 5, we find useful to state two lemmas.

Lemma 6. *Let F be a map such that $\log F$ is concave. Then, for every $n_1, \ldots, n_k$,*

$$\prod_{i=1}^{k} F(n_i) \leq F\left(\frac{\sum_{i=1}^{k} n_i}{k}\right)^k. \tag{3.1}$$

In particular, if F is subexponential, then there is a map $G \geq F$ such that $\log G$ is concave, and hence G satisfies equation (3.1).

Proof. By hypothesis, $\sum_{i=1}^{k} \log F(n_i) \leq k \log F\left(\frac{\sum_{i=1}^{k} n_i}{k}\right)$. Exponentiating this last equation, the desired inequality follows.

Suppose now that F is subexponential, that is, $\lim_{n\to\infty} \frac{\log F(n)}{n} = 0$. Let $(\varepsilon_i)_{i\geq 1}$ be strictly decreasing to zero and $(n_i)_{i\geq 1}$ be strictly increasing, such that $n_1 = 1$ and $\frac{\log F(n)}{n} \leq \varepsilon_i$ for every $n \geq n_i$. . Define then $\log G(n) = \varepsilon_i n + \delta_i$ on the the interval $n_i \leq n \leq n_{i+1}$, with $\delta_1 = 0$ and $\delta_i = (\varepsilon_{i-1} - \varepsilon_i)n_i + \delta_{i-1}$. Then $\log G \geq \log F$ is continuous and concave by definition and $\lim_{n\to\infty} \frac{\log G(n)}{n} = 0$. ∎

Lemma 7. *Consider the maps*

$$\lambda(n) = \frac{n \log\log n}{\log n}$$

and for some $d, m > 0$,

$$f(n) = \frac{\log n}{n\left(\log\log n\right)^2} + \frac{d^m(\log n)^2}{n(\log\log n)^2} + \frac{n-\lambda(n)}{n}\frac{\log n}{\log\left(\frac{n-\lambda(n)}{d^m}\right)}\left(\frac{\log\log\left(\frac{n-\lambda(n)}{d^m}\right)}{\log\log n}\right)^2.$$

Then, there exists an integer N such that $f(n) \leq 1$ for every $n \geq N$.

Proof. We write

$$f(n) = \frac{1}{\log\log(n)^2}\cdot\frac{\log n}{n}(1 + A\log n) + \frac{\log\log(n')^2}{\log\log(n)^2}\cdot A\frac{n'\log n}{n\log(n')},$$

with $A = d^m$ and $n' = (n-\lambda(n))/A$. Since $\frac{1}{\log\log(n)^2} < 1$ and $\frac{\log\log(n')^2}{\log\log(n)^2} < 1$ for n large enough, it suffices to prove the stronger inequality

$$\frac{\log n}{n}(1 + A\log n) + A\frac{n'\log n}{n\log n'} < 1 \tag{3.2}$$

for all n large enough.

Now this amounts to

$$\frac{\log n}{n}(1 + A\log n) < 1 - \frac{\log n}{\log n'} + \frac{\log\log n}{\log n'};$$

if we multiply this last inequality by $\frac{\log n'}{\log\log n}$, we get

$$\frac{\log n\log n'}{n\log\log n}(1 + A\log n) < 1 - \frac{\log(n/n')}{\log\log n}.$$

Then the LHS is bounded above by $(A+1)\frac{\log(n)^3}{n\log\log n}$, which tends to 0 as $n \to \infty$; and $\frac{\log(n/n')}{\log\log n}$ also tends to 0 as $n \to \infty$ because n/n' tends to A, so the RHS tends to 1. It follows that (3.2) holds for n large enough. ∎

Proof of Proposition 5. We first suppose that $\mathcal{I}_k$ has subexponential growth, for some k. What will actually be used is that $\mathcal{I}$ itself has subexponential growth.

Let us write every $g \in G$ as a product $g = g_1 \cdots g_{N(g)}$ with $g_i \in \mathcal{I}$ and where $N(g) = \min\{k \mid g = g_1 \cdots g_k,\ g_i \in \mathcal{I}\}$.

For any $\lambda \leq \frac{n}{2}$, the sphere of ray n in G is the union of

$$W_\lambda^<(n) := \{g \mid \ell(g) = n, N(g) \leq \lambda\} \quad\text{and}\quad W_\lambda^>(n) := \{g \mid \ell(g) = n, N(g) > \lambda\}.$$

Let $\mathcal{I}(n_i)$ be the sphere of ray n_i in $\mathcal{I}$ and $\delta(n_i) = \#\mathcal{I}(n_i)$. Then, for any $k \leq \lambda$, the cardinality of $\mathcal{I}^k \cap \{g \in G \mid \ell(g) = n\}$ is $\sum_{n_1+\cdots+n_k=n}\prod_{i=1}^k \delta(n_i)$. Hence,

$$\#\, W_\lambda^<(n) \leq \sum_{k=1}^{\lambda}\sum_{n_1+\cdots+n_k=n}\prod_{i=1}^{k}\delta(n_i).$$

We may suppose that $\delta(n)$ is increasing and, by Lemma 6, satisfies equation (3.1). Hence,

$$\begin{aligned}\#\, W_\lambda^<(n) &\leq \sum_{k=1}^{\lambda}\sum_{n_1+\cdots+n_k=n}\delta\left(\frac{n}{k}\right)^k \\ &\leq \sum_{k=1}^{\lambda}\sum_{\binom{n-1}{k-1}}\delta\left(\frac{n}{\lambda}\right)^\lambda \leq \lambda\binom{n-1}{\lambda-1}\delta\left(\frac{n}{\lambda}\right)^\lambda.\end{aligned}$$

As $\binom{n}{\lambda} \leq (\frac{en}{\lambda})^\lambda$ by Stirling's formula, it follows that

$$\# W_\lambda^<(n) \leq e^\lambda \left(\frac{n}{\lambda}\right)^{\lambda-1} \delta\left(\frac{n}{\lambda}\right)^\lambda . \tag{3.3}$$

On the other hand, for n fixed, there is an m such that $\mathcal{I}_m(n) = \mathcal{I}(n)$. Therefore,

$$\# W_\lambda^>(n) \leq [G : Stab(m)] \sum_{n_1+\ldots+n_{d^m} \leq n-\lambda} \gamma(n_1)\cdots\gamma(n_{d^m}).$$

If $\gamma = \lim_{n\to\infty} \gamma(n)^{1/n}$ is the growth rate of G, then there is a constant $K > 0$ such that $K\gamma \geq \gamma(n)^{1/n}$ for every $n \geq 1$. Hence,

$$\# W_\lambda^>(n) \leq [G : Stab(m)] \sum_{n_1+\ldots+n_{d^m} \leq n-\lambda} K^{d^m} \gamma^{n-\lambda}$$

and

$$\# W_\lambda^>(n) \leq p(n)\gamma^{n-\lambda} \tag{3.4}$$

where $p(n) = [G : Stab(m)]\, K^{d^m} \binom{n-\lambda}{d^m}$ is a polynomial (of degree d^m).

Set $\varepsilon = \frac{\lambda}{n}$. From equations (3.3) and (3.4) we get

$$\begin{aligned}\gamma &\leq \lim_{n\to\infty} \left(\# W_\lambda^>(n) + \# W_\lambda^<(n)\right)^{1/n} \\ &\leq \max\left\{\lim_{n\to\infty} \# W_\lambda^>(n)^{1/n}, \lim_{n\to\infty} \# W_\lambda^<(n)^{1/n}\right\} \leq \max\left\{\varepsilon^{-\varepsilon}\delta(\varepsilon^{-1})^\varepsilon, \gamma^{1-\varepsilon}\right\}.\end{aligned}$$

As $\lim_{\varepsilon\to 0} \varepsilon^{-\varepsilon}\delta(\varepsilon^{-1})^\varepsilon = 1$, obtain in all cases $\gamma = 1$.

Suppose next that I_k grows linearly for some k. Thus, there is an $m(\geq k)$ such that $\mathcal{I}_m = \mathcal{I}$. We have to show that there exist constants $A, B > 0$ such that

$$\gamma(n) \leq \exp\left(A + B\frac{n(\log\log n)^2}{\log n}\right),$$

for n large enough.

Consider the subexponential map $F(n) = e^{\frac{n(\log\log n)^2}{\log n}}$. Then, for $n \geq c := e^{e^2}$,

$$\begin{aligned}(\log F(n))'' = \frac{1}{n(\log n)^3}\big(&-\log n(\log\log n)^2 + 2\log n\log\log n \\ &+ 2(\log\log n)^2 - 6\log\log n + 2\big) \leq 0\end{aligned}$$

so that $\log F(n)$ is concave for $n \geq c$.

Define $A = \log\gamma(N)$, where N is as in Lemma 7. Consider also the constants

$$\begin{aligned}M &= (d^m+1)[G : Stab(m)]\gamma(c)^{d^m}\left(\frac{e}{d^m}\right)^{d^m} \\ \text{and}\quad B &= \max\left\{2 + \log\delta\left(\frac{n}{\lambda}\right), \log M + (d^m-1)A + \log 2\right\}.\end{aligned}$$

Define then the map

$$F(n) = \begin{cases} \exp\left(A + B\frac{n(\log\log n)^2}{\log n}\right) & \text{if} \quad n \geq c \\ \exp\left(A + B\frac{c(\log\log c)^2}{\log c}\right) & \text{if} \quad 0 \leq n < c, \end{cases}$$

so that $\gamma(k) \leq F(k)$ for every $k \leq N$. For $n > N$, let us show by induction that $\gamma(n) \leq F(n)$.

As before, since $\mathcal{I} = \mathcal{I}_m$, we have

$$\begin{aligned}\# W_\lambda^>(n) &\leq [G : Stab(m)] \sum_{n_1+\ldots+n_{d^m} \leq n-\lambda} \gamma(n_1)\cdots\gamma(n_{d^m}) \\ &\leq [G : Stab(m)] \sum_{n_1+\ldots+n_{d^m} \leq n-\lambda} F(n_1)\cdots F(n_{d^m}).\end{aligned}$$

Developing this last sum and thanks to Lemma 6, we get

$$\# W_\lambda^>(n) \leq [G : Stab(m)] \binom{n-\lambda}{d^m} \left(F(c)^{d^m} + \sum_{k=1}^{d^m} F(c)^{d^m-k} F\left(\frac{n-\lambda}{k}\right)^k \right).$$

Hence,

$$\# W_\lambda^>(n) \leq (d^m+1)\, [G : Stab(m)]\, F(c)^{d^m} \binom{n-\lambda}{d^m} F\left(\frac{n-\lambda}{d^m}\right)^{d^m}.$$

Thus, as $\binom{n-\lambda}{d^m} \leq \left(\frac{e(n-\lambda)}{d^m}\right)^{d^m} < \left(\frac{e}{d^m}\right)^{d^m} n^{d^m}$, we get

$$\# W_\lambda^>(n) \leq M\, n^{d^m}\, F\left(\frac{n-\lambda}{d^m}\right)^{d^m}.$$

Together with (3.3), this gives

$$\gamma(n) \leq \# W_\lambda^<(n) + \# W_\lambda^>(n) \leq \left(\frac{n}{\lambda}\right)^{\lambda-1} \delta \left(\frac{n}{\lambda}\right)^{\lambda} + Mn^{d^m} F\left(\frac{n-\lambda}{d^m}\right)^{d^m}.$$

For $\lambda = \frac{n\log\log n}{\log n}$, we see that $(\lambda-1)\log\left(\frac{n}{\lambda}\right) + \lambda\log\delta\left(\frac{n}{\lambda}\right) \leq A - \log 2 + B\frac{n(\log\log n)^2}{\log n}$, and hence

$$\left(\frac{n}{\lambda}\right)^{\lambda-1} \delta \left(\frac{n}{\lambda}\right)^{\lambda} \leq \frac{1}{2}F(n).$$

It remains to verify that

$$Mn^{d^m} F\left(\frac{n-\lambda}{d^m}\right)^{d^m} \leq \frac{1}{2}F(n).$$

But this is equivalent to

$$\begin{aligned}&\frac{(\log M + (d^m-1)A + \log 2)\log n}{Bn(\log\log n)^2} + \frac{d^m(\log n)^2}{Bn(\log\log n)^2} \\ &\qquad + \frac{n-\lambda}{n}\frac{\log n}{\log\left(\frac{n-\lambda}{d^m}\right)} \left(\frac{\log\log\left(\frac{n-\lambda}{d^m}\right)}{\log\log n}\right)^2 \leq 1. \qquad (3.5)\end{aligned}$$

As the left side of (3.5) is smaller than $f(n)$ by definition of B, this holds by Lemma 7. ∎

4. Growth of the Fabrykowski-Gupta group

In the remainder, G will denote the Fabrykowski-Gupta group, as defined in Section 2.

4.1. **Proof of Theorem 1.** The lower bound is easily computed. Indeed, consider the morphism $\psi\colon G' \to G'$ induced by $a \mapsto t$ and $t \mapsto t^a$, where $G' = \left\langle t^{\pm a^i} t^{\mp a^j},\, i \neq j \right\rangle$. Since $\psi\left(t^{\pm a^i} t^{\mp a^j}\right) = \left\langle\left\langle t^{\pm a^i} t^{\mp a^j}, 1, 1 \right\rangle\right\rangle$, there is an injective map

$$\left(B_G(n) \cap G'\right)^3 \hookrightarrow B_G(6n) \cap G' \;;\; (g_1, g_2, g_3) \mapsto \psi(g_1)\psi(g_2)^a \psi(g_3)^{a^2}$$

where $B_G(n)$ is the ball of radius n in G. Hence, $\beta(6n) \geq \beta(n)^3$, with $\beta(n) = \#\left(B_G(n) \cap G'\right)$. Iterating this inequality, one get $\beta(2 \cdot 6^n) \geq \beta(2)^{3^n} = 12^{3^n}$, so that

$$\gamma(t) \geq \beta(t) \geq 12^{(t/2)^{\frac{\log 3}{\log 6}}}.$$

On the other hand, the upper bound follows directly from the following result and Proposition 5.

Proposition 8. (1) *If $w \notin \mathcal{I}$, then w has length reduction up to the third level. Equivalently, $\mathcal{I} = \mathcal{I}_3$;*
(2) *The growth of $\mathcal{I}$ is linear.*

Before we prove Proposition 8, let us give some definitions and lemmas.

4.2. **Length reduction of words.** Consider the subsets of A^*

$$\mathcal{S} = \{s | s \text{ is a subword of } (\ldots 0\,2\,1\,0\,1\,2\,0 \ldots)^\sigma, \text{ for } \sigma \in A\}$$

and

$$\partial\mathcal{S} = \{s | s \text{ is a subword of } (\ldots 1\,1\,1\,2\,2\,2 \ldots)\}.$$

Note that

$$s = (s_i)_{i=1}^n \in A\partial\mathcal{S} \text{ if and only if } \Sigma s := \left(-\sum_{k=1}^{i} s_k\right)_{i=1}^{n} \in \mathcal{S}. \tag{4.1}$$

For sequences $c = (c_i)_{i=1}^n \in \partial\mathcal{S}$ and $\gamma = (\gamma_i)_{i=1}^n \in S$, consider the maps

$$m(c) = \begin{cases} 1 & \text{if } c \text{ is a subword of } (012)^\infty \\ k & \text{if } c_{k-1} = c_{k+1} \\ n & \text{if } c \text{ is a subword of } (021)^\infty \end{cases}$$

and

$$\partial m(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is a subword of } 2^\infty \\ k & \text{if } \gamma_k = 1 \text{ and } \gamma_{k+1} = 2 \\ n & \text{if } \gamma \text{ is a subword of } 1^\infty \end{cases}$$

so that, obviously, $\partial m(\gamma) = m(\Sigma\gamma)$.

Define next, for an element g written as in (2.1), the *exponent sequence* $\gamma(g) = (\gamma_i)_{i=1}^n$ and the *index sequence* $c(g) = (c_i)_{i=1}^n$ of g.

As $t_{\sigma(i)} = t_i^\sigma$ for any $i, \sigma \in A$, the following remark holds.

Lemma 9. *Let $s = t_{c_1}^{\gamma_1} \cdots t_{c_n}^{\gamma_n}\tau = \langle\langle\, s_0, s_1, s_2 \,\rangle\rangle\tau$ be any word and its first level decomposition, and let $\sigma \in A$. Then*

$$s_\sigma := t_{\sigma(c_1)}^{\gamma_1} \cdots t_{\sigma(c_n)}^{\gamma_n}\tau = \langle\langle\, s_0, s_1, s_2 \,\rangle\rangle^\sigma\tau.$$

In particular, s and s_σ have the same first level decompositions up to a permutation of the components.

Recall that $\mathcal{I}_1 = \{g \in G | \sum_{i \in A} \ell(g_i) = \ell(g)\}$. It is characterized in the following way.

Lemma 10. *The set $\mathcal{I}_1$ is exactly the set of elements g that may be written as*

$$g = t_{c_1}^{\gamma_1} \cdots t_{c_{m-1}}^{\gamma_{m-1}} t_{c_m}^{\gamma_m} t_{c_{m+1}}^{\gamma_{m+1}} \cdots t_{c_n}^{\gamma_n} \tau$$

with $\gamma_i = \pm 1$, and $c_i \in A$ such that

(a) *$c(g) \in \mathcal{S}$ (with, say, $m(c(g)) = m$),*

(b) *If $2 < m < n-2$, then $\gamma_{m-1} = \gamma_{m+1}$.*

Proof. Suppose that g satisfies (a) and (b). If $m = 1$ or $m = n$, then g is obviously in $\mathcal{I}_1$. Otherwise, g contains a subword

$$\begin{aligned} s = t_{\sigma(1)}^{\alpha} t_{\sigma(0)}^{\beta} t_{\sigma(1)}^{\gamma} &= \langle\langle\, t^\alpha, a^\alpha, 1 \,\rangle\rangle^\sigma \langle\langle\, a^\beta, 1, t^\beta \,\rangle\rangle^\sigma \langle\langle\, t^\gamma, a^\gamma, 1 \,\rangle\rangle^\sigma \\ &= \langle\langle\, t^\alpha a^\beta t^\gamma, a^\alpha a^\gamma, t^\beta \,\rangle\rangle^\sigma, \end{aligned}$$

where $\sigma \in A$. If $2 < m < n-2$ and $\alpha = \gamma$ then $s = \langle\langle\, t^\alpha a^\beta t^\alpha, a^{2\alpha}, t^\beta \,\rangle\rangle^\sigma$ so that $\sum_{i \in A} \ell(g_i) = \ell(g)$.

Reciprocally, suppose that g does not satisfy (a). Then g contains a subword

$$\begin{aligned} u = t_{\sigma(0)}^{\alpha} t_{\sigma(1)}^{\beta} t_{\sigma(0)}^{\gamma} &= \langle\langle\, a^\alpha, 1, t^\alpha \,\rangle\rangle^\sigma \langle\langle\, t^\beta, a^\beta, 1 \,\rangle\rangle^\sigma \langle\langle\, a^\gamma, 1, t^\gamma \,\rangle\rangle^\sigma \\ &= \langle\langle\, a^\alpha t^\beta a^\gamma, a^\beta, t^{\alpha-\gamma} \,\rangle\rangle^\sigma, \end{aligned}$$

We see that $\sum_{i \in A} \ell(s_i) \le \ell(s) - 1 < \ell(s)$, hence $\sum_{i \in A} \ell(g_i) < \ell(g)$.

Finally, suppose that g does not satisfy (b), that is, $2 < m < n-2$ and g contains a subword

$$s = t_{\sigma(1)}^{\alpha} t_{\sigma(0)}^{\beta} t_{\sigma(1)}^{-\alpha} = \langle\langle\, t^\alpha a^\beta t^\alpha, 1, t^\beta \,\rangle\rangle^\sigma.$$

Again, $\sum_{i \in A} \ell(g_i) < \ell(g)$. ∎

Let $g = t_{c_1}^{\gamma_1} t_{c_2}^{\gamma_2} \cdots t_{c_{m-1}}^{\gamma_{m-1}} t_{c_m}^{\gamma_m} t_{c_{m+1}}^{\gamma_{m+1}} \cdots t_{c_n}^{\gamma_n} \tau$ be a group element with $c(g) \in \mathcal{S}$ and $m(c(g)) = m$. Developing g on the first level, we get

$$\begin{aligned} \langle\langle\, g_0, g_1, g_2 \,\rangle\rangle = \cdots \langle\langle\, 1, t^{\gamma_{m-2}}, a^{\gamma_{m-2}} \,\rangle\rangle^{a^{c_m}} \langle\langle\, t^{\gamma_{m-1}}, a^{\gamma_{m-1}}, 1 \,\rangle\rangle^{a^{c_m}} \langle\langle\, a^{\gamma_m}, 1, t^{\gamma_m} \,\rangle\rangle^{a^{c_m}} \\ \langle\langle\, t^{\gamma_{m+1}}, a^{\gamma_{m+1}}, 1 \,\rangle\rangle^{a^{c_m}} \langle\langle\, 1, t^{\gamma_{m+2}}, a^{\gamma_{m+2}} \,\rangle\rangle^{a^{c_m}} \cdots . \end{aligned}$$

By Lemma 9, up to a permutation of the components, we may suppose $c_m = 0$. We may also suppose that $c_1 = 1$, as the two remaining cases behave symmetrically. Hence we get

$$\begin{aligned} g_0 &= t^{\gamma_1} a^{\gamma_2} t^{\gamma_4} a^{\gamma_5} \cdots a^{\gamma_{m-3}} t^{\gamma_{m-1}} a^{\gamma_m} t^{\gamma_{m+1}} a^{\gamma_{m+3}} t^{\gamma_{m+4}} \cdots \\ &= t_0^{\gamma_1} t_{-\gamma_2}^{\gamma_4} \cdots t_*^{\gamma_{m-4}} t_{*-\gamma_{m-3}}^{\gamma_{m-1}} t_{*-\gamma_{m-3}-\gamma_m}^{\gamma_{m+1}} t_{*-\gamma_{m-3}-\gamma_m-\gamma_{m+3}}^{\gamma_{m+4}} \cdots \\ g_1 &= a^{\gamma_1} t^{\gamma_3} a^{\gamma_4} t^{\gamma_6} \cdots a^{\gamma_{m-4}} t^{\gamma_{m-2}} a^{\gamma_{m-1}+\gamma_{m+1}} t^{\gamma_{m+2}} a^{\gamma_{m+4}} t^{\gamma_{m+5}} \cdots \\ &= t_{-\gamma_1}^{\gamma_3} t_{-\gamma_1-\gamma_4}^{\gamma_6} \cdots t_*^{\gamma_{m-5}} t_{*-\gamma_{m-4}}^{\gamma_{m-2}} t_{*-\gamma_{m-4}-(\gamma_{m-1}+\gamma_{m+1})}^{\gamma_{m+2}} \\ &\qquad t_{*-\gamma_{m-4}-(\gamma_{m-1}+\gamma_{m+1})-\gamma_{m+4}}^{\gamma_{m+5}} \cdots \\ g_2 &= t^{\gamma_2} a^{\gamma_3} t^{\gamma_5} a^{\gamma_6} \cdots a^{\gamma_{m-5}} t^{\gamma_{m-3}} a^{\gamma_{m-2}} t^{\gamma_m} a^{\gamma_{m+2}} t^{\gamma_{m+3}} \cdots \\ &= t_0^{\gamma_2} t_{-\gamma_3}^{\gamma_5} \cdots t_*^{\gamma_{m-6}} t_{*-\gamma_{m-5}}^{\gamma_{m-3}} t_{*-\gamma_{m-5}-\gamma_{m-2}}^{\gamma_m} t_{*-\gamma_{m-5}-\gamma_{m-2}-\gamma_{m+2}}^{\gamma_{m+3}} \cdots . \end{aligned}$$

Set $\tilde\gamma(g_0) = (\gamma_1,\ \gamma_4, \ldots, \gamma_{m-4},\ \gamma_{m-1} + \gamma_{m+1},\ \gamma_{m+4}, \ldots)$. If $\tilde\gamma(g_0), \gamma(g_1), \gamma(g_2) \in A\partial\mathcal{S}$, then, thanks to (4.1), the following relations hold

$$\partial m(\gamma(g_1)) + 1 = m(c(g_2)), \tag{4.2}$$

$$\partial m(\gamma(g_2)) + 1 = m(c(g_0)), \tag{4.3}$$

$$\partial m(\tilde\gamma(g_0)) = m(c(g_1)). \tag{4.4}$$

Lemma 11. (1) *$g \in \mathcal{I}_n$ if and only if $g \in \mathcal{I}_{n-1}$ and $g_x \in \mathcal{I}_1$ for every $g \in A^{n-1}$;*

(2) *For every* $x \in A^*$,

$$\forall i \in A\colon c(g_{xi}) \in \mathcal{S} \text{ if and only if } \forall i \in A^\times : \gamma(g_{xi}) \in A\partial\mathcal{S} \text{ and } \tilde{\gamma}(g_{x0}) \in A\partial\mathcal{S};$$

(3) $g \in \mathcal{I}$ *if and only if* $c(g_x) \in \mathcal{S}$ *for every* $x \in A^*$;

(4) $g \in \mathcal{I}$ *if and only if* $\gamma(g_{xi}) \in A\partial\mathcal{S}$ *for* $i \in A^\times$ *and* $\tilde{\gamma}(g_{x0}) \in A\partial\mathcal{S}$ *for every* $x \in A^*$.

Proof. (1) This follows from the definition.

(2) Applying (4.1) to the exponent sequences of the components of g_x, equations (4.2)-(4.4) show that $\tilde{\gamma}(g_{x0}) \in A\partial\mathcal{S}$ if and only if $c(g_{x1}) \in \mathcal{S}$, that $\gamma(g_{x1}) \in A\partial\mathcal{S}$ if and only if $c(g_{x2}) \in \mathcal{S}$ and that $\gamma(g_{x2}) \in A\partial\mathcal{S}$ if and only if $c(g_{x0}) \in \mathcal{S}$.

(3) If there exists $x \in A^*$ such that $c(g_x) \notin \mathcal{S}$, then $g_x \notin \mathcal{I}_1$ by Lemma 10, so that $g \notin \mathcal{I}$. Reciprocally, fix $x \in A^*$ and write $g_x = t_{c_1}^{\gamma_1} \cdots t_{c_m}^{\gamma_m} \cdots t_{c_n}^{\gamma_n}\tau$, with $m(c(g_x)) = m$. By hypothesis, $c(g_x) \in S$, so that (by (1)) it is enough to see that, if $2 < m < n-2$, then $\gamma_{m-1} = \gamma_{m+1}$. But $c(g_{x1}) \in \mathcal{S}$ so that $\gamma(g_{x0}) \in A\partial\mathcal{S}$, by (2). Therefore, $\gamma_{m-1} = \gamma_{m+1}$.

(4) This follows from (2) and (3). ∎

Lemma 12. *Let* $g = t_{c_1}^{\gamma_1} \cdots t_{c_m}^{\gamma_m} \cdots t_{c_n}^{\gamma_n}\tau$ *be an element of* I_1 *of length* n, *with* $m(c(g)) = m$, *and such that* $\gamma(g) \in \partial\mathcal{S}$. *Suppose moreover that* $10 < m < n - 10$. *Then* $g \notin \mathcal{I}$.

Proof. If $\gamma_{m-1} \neq \gamma_{m+1}$, then $g \notin \mathcal{I}_1$ by Lemma 10. Also, if $\tilde{\gamma_0}(g) \notin \partial\mathcal{S}$, then $c(g_1) \notin \mathcal{S}$, so $g_1 \notin \mathcal{I}_1$. Suppose now that $\gamma_{m-1} = \gamma_{m+1}$ and $\tilde{\gamma_0}(g) \in \partial\mathcal{S}$. As $\gamma_0(g) \in \partial\mathcal{S}$ by hypothesis, we have $\partial m(\gamma(g_0)) \in \{\frac{m-2}{3}, \frac{m+4}{3}\}$. Thus, there are 6 remaining choices for $\partial m(\gamma(g))$:

- $\partial m(\gamma(g)) = m+1$ or $\partial m(\gamma(g)) = m+2$. In those cases, $\gamma_{m-4} = 1 = \gamma_{m+1}$ and $\gamma_{m+4} = 2$. But $\partial m(\gamma(g_2)) = \frac{m+1}{3}$, so that $m(c(g_0)) = \frac{m+4}{3}$.
- $\partial m(\gamma(g)) = m+3$. In that case, $\gamma_{m+1} = 1$ and $\gamma_{m+4} = 2 = \gamma_{m+7}$. But $\partial m(\gamma(g_2)) = \frac{m+4}{3}$, so that $m(c(g_0)) = \frac{m+7}{3}$.
- $\partial m(\gamma(g)) = m-2$ or $\partial m(\gamma(g)) = m-3$. In those cases, we have $\gamma_{m-4} = 1$ and $\gamma_{m-1} = 2 = \gamma_{m+1}$. But $\partial m(\gamma(g_2)) = \frac{m-2}{3}$, so that $m(c(g_0)) = \frac{m+1}{3}$.
- $\partial m(\gamma(g)) = m-4$. In that case, $\gamma_{m-7} = 1 = \gamma_{m-4}$ and $\gamma_{m-1} = 2$. But $\partial m(\gamma(g_2)) = \frac{m-5}{3}$, so that $m(c(g_0)) = \frac{m-2}{3}$.

In any of those cases, using Lemma 10, we see that $g_0 \notin \mathcal{I}_1$, so that g does not belong to $\mathcal{I}$. ∎

4.3. **Proof of Proposition 8.**

(1) Let $g = t_{c_1}^{\gamma_1} \cdots t_{c_m}^{\gamma_m} \cdots t_{c_n}^{\gamma_n}\tau \in \mathcal{I}_3$, with $m(c(g)) = m$. For every $i \in A$, we know by hypothesis that $\gamma(g_i), \gamma(g_{ij}) \in A\partial\mathcal{S}$ for $j \neq 0$ and $\tilde{\gamma}(g_0), \tilde{\gamma}(g_{i0}) \in A\partial\mathcal{S}$.

By Lemma 11 (4), all we have to show is that $\gamma(g_{xi}) \in A\partial\mathcal{S}$ for $i \in A^\times$ and $\tilde{\gamma}(g_{x0}) \in A\partial\mathcal{S}$, for every $x \in A^2A^*$. Now,

- For $j \in A^\times$, as $\gamma(g_{0j}) \in A\partial\mathcal{S}$, the index sequence $\gamma(g_0)$ is of one of the following types

$\cdots$	γ_{m-7}	γ_{m-4}	γ_{m-1}	γ_{m+1}	γ_{m+4}	γ_{m+7}	$\cdots$
...	1	2	1	1	2	2	...
...	1	1	1	1	2	2	...
...	1	1	2	2	2	2	...
...	1	1	2	2	1	2	...

which means that

$$\partial m\left(\tilde{\gamma}(g_0)\right) \in \left\{\frac{m-5}{3},\ \frac{m-2}{3},\ \frac{m+1}{3},\ \frac{m+4}{3}\right\}. \tag{4.5}$$

In any of those cases, note that we also have $\gamma(g_{00}) \in A\partial\mathcal{S}$. Altogether, this implies that $\gamma(g_{0y}) \in A\partial\mathcal{S}$ for every $y \in A^*$. Hence, $\gamma(g_{0xi}) \in A\partial\mathcal{S}$ for $i \in A^\times$ and $\tilde{\gamma}(g_{0x0}) \in A\partial\mathcal{S}$ for every $x \in A^*$.

- For $i \in A^\times$, since $\gamma(g_i) \in A\partial\mathcal{S}$, we have $\gamma(g_{iy}) \in A\partial\mathcal{S}$ for every $y \in A^*$. Hence, $\gamma(g_{ixj}) \in A\partial\mathcal{S}$ for $j \in A^\times$ and $\tilde{\gamma}(g_{ix0}) \in A\partial\mathcal{S}$ for every $x \in A^*$. Moreover, $\gamma(g_{ij}) \in A\partial\mathcal{S}$ for $i \in A^\times$ and $j \in A$ implies that

$$m(c(g_1)) \in \{\partial m(\gamma(g_1)) - 1,\ \partial m(\gamma(g_1)) \pm 2,\ \partial m(\gamma(g_1)) \pm 3, \partial m(\gamma(g_1)) + 4\}, \tag{4.6}$$

$$m(c(g_2)) \in \{\partial m(\gamma(g_2)) - 1,\ \partial m(\gamma(g_2)) \pm 2,\ \partial m(\gamma(g_2)) \pm 3, \partial m(\gamma(g_1)) + 4\}. \tag{4.7}$$

Using relations (4.2)-(4.4) and (4.5),(4.6) and (4.7), we see that, given one of $m(c(g))$, $m(c(g_0))$, $m(c(g_1))$ or $m(c(g_2))$, the number of possibilities of choosing the three others (so that g remains in $\mathcal{I}$) is bounded by a constant (independently of the length of g).

(2) We have to show that $\delta(n) = \#\,\mathcal{I}(n)$ is bounded by a constant (independent of n). But

$$\delta(n) \leq K\ \#\{\text{ possible choices of } m(c(g))\}\ \#\{\text{ possible choices of } m(c(g_i)), i \in A\}.$$

Let $g \in \mathcal{I}$. For $i \in A^\times$, we know by Lemma 11 (4) that $\gamma(g_i) \in A\partial\mathcal{S}$. Hence, by Lemma 12, we have $m(c(g_i)) \leq 10$ or $m(c(g_i)) \geq n-10$. Therefore, there is at most 20 choices for $m(c(g_i))$ (to be chosen between 1 and n). Now, the last assertion in the proof of (1) insures that the remaining choices of m and $m(c(g_0))$ are bounded by a constant. ∎

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